The Gamma Distribution

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The Gamma Distribution is a distribution of waiting times between Poisson distributed events. In the white paper [The Gamma Function] we developed the mathematics for the gamma function, which is a function that is related to the factorial function and is the basis for the Gamma Distribution. In this white paper we will derive the mathematics for Gamma Distribution. To that end we will use the following hypothetical problem...

Our Hypothetical Problem

You are tasked with purchasing 200 BB rated bonds for inclusion in a CDO (Collateralized Debt Obligation). Using Moody's Corporate Bond Default and Recovery data (see table below) what is the probability that 10 or more bonds default in any given year given that none of the 200 bonds have defaulted at the beginning of that year?

Table 1: Moody's BB Rated Bond Average Default Rate 1920 to 2010

Default rate mean	1.070%
Default rate standard deviation	1.612%

Note: For this exercise we can ignore the correlation effect of selecting bonds from only one asset class.

The Probability Density Function

In the white paper [The Gamma Function] we defined the gamma function of the variable α to be the following equation...

$$\Gamma(\alpha)=\int\limits_{u=0}^{u=\infty} u^{\alpha-1}\,\mathrm{Exp}\left\{-u\right\}\delta u \;\text{ ...where... }\; \{\alpha\,|\,\alpha\in\mathbb{R},\alpha>0\} \tag{1}$$

The first step in developing the mathematics for the Gamma Distribution will be to redefine the integrand u in the gamma function as defined by Equation (1) above via the following change of variables...

$$\text{if... } u=\lambda\,x \text{ ...then... } x=\frac{1}{\lambda}\,u \text{ ...and... } \frac{\delta u}{\delta x}=\lambda \text{ ..and... } \delta u=\lambda\,\delta x \tag{2}$$

The next step will be to change the integrand in Equation (1) above from u to x. Using the definitions in Equation(2) above the revised bounds of integration for the new integrand x are...

$$\text{Upper bound x} = \frac{1}{\lambda}\times\infty=\infty \text{ ...and... Lower bound x} = \frac{1}{\lambda}\times 0 = 0 \text{ ...given that... } x=\frac{1}{\lambda}\,u \tag{3}$$

Using Equations (2) and (3) above we can rewrite Equation (1) above as...

$$\Gamma(\alpha)=\int\limits_{x=0}^{x=\infty}(\lambda\,x)^{\alpha-1}\,\mathrm{Exp}\left\{-(\lambda\,x)\right\}(\lambda\,\delta x)=\int\limits_{x=0}^{x=\infty}\lambda^{\alpha}\,x^{\alpha-1}\,\mathrm{Exp}\left\{-\lambda\,x\right\}\delta x \tag{4}$$

In Equation (4) above the parameter α is referred to as the **shape** parameter and the parameter λ is referred to as the **scale** parameter. Rather than use the scale parameter most textbooks use the **rate** parameter θ, which is the inverse of the scale parameter. Using these revised parameter definitions we can rewrite Equation (4) above as...

$$\Gamma(\alpha)=\int\limits_{x=0}^{x=\infty}\theta^{-\alpha}\,x^{\alpha-1}\,\mathrm{Exp}\left\{-x\,\theta^{-1}\right\}\delta x \;\text{ ...where... }\; \theta=\frac{1}{\lambda} \tag{5}$$

We will define the function $f(\alpha,\theta)$ to be the probability density function (PDF) for the Gamma Distribution. Using Equation (5) above the equation for the PDF is...

$$f(\alpha,\theta)=\frac{x^{\alpha-1}\,\mathrm{Exp}\left\{-x\,\theta^{-1}\right\}}{\theta^{\alpha}\Gamma(\alpha)}\ \text{...noting that...}\ \int\limits_{x=0}^{x=\infty} f(\alpha,\theta)\,\delta x=\frac{\Gamma(\alpha)}{\Gamma(\alpha)}=1 \tag{6}$$

The Moment Generating Function

Using Appendix Equation (25) below the equation for the moment generating function (MGF) of the Gamma Distribution is...

$$M_x(t)=\frac{1}{\theta^{\alpha}\Gamma(\alpha)}\int\limits_{x=0}^{x=\infty} x^{\alpha-1}\,\mathrm{Exp}\left\{-x\left(\theta^{-1}-t\right)\right\}\delta x \tag{7}$$

We will change integrand x in the MGF as defined by Equation (7) above via the following change of variables...

$$y=x\left(\theta^{-1}-t\right)\ \text{...such that...}\ x=\frac{\theta}{1-\theta\,t}\,y\ \text{...and...}\ \frac{\delta x}{\delta y}=\frac{\theta}{1-\theta\,t}\ \text{...and...}\ \delta x=\frac{\theta}{1-\theta\,t}\,\delta y \tag{8}$$

When changing the integrand in Equation (7) above from x to y the bounds of integration will change. Using the definitions in Equation(8) above the revised bounds of integration for the new integrand y are...

$$\text{Upper bound y}=\left(\theta^{-1}-t\right)\times\infty=\infty\ \text{...and...}\ \text{Lower bound y}=\left(\theta^{-1}-t\right)\times 0=0\ \text{...since...}\ y=x\left(\theta^{-1}-t\right) \tag{9}$$

Using Appendix Equation (26) below the equation for the moment generating function of the Gamma Distribution post change of variables is...

$$M_y(t)=\left(1-\theta\,t\right)^{-\alpha} \tag{10}$$

For ease of calculating the moments of the Gamma Distribution we will redefine the moment generating function as defined by Equation (10) above as follows...

$$M_y(t)=\omega^{-\alpha}\ \text{...where...}\ \omega=1-\theta\,t\ \text{...and...}\ \frac{\delta}{\delta t}\,\omega=-\theta \tag{11}$$

Using Equation (11) above the equation for the first derivative of the moment generating function with respect to the variable t is...

$$\frac{\delta}{\delta t}\,M_y(t)=\frac{\delta}{\delta\omega}\,M_y(t)\,\frac{\delta}{\delta t}\,\omega=-\alpha\,\omega^{-\alpha-1}\times-\theta=\alpha\,\theta\,\omega^{-\alpha-1} \tag{12}$$

Using Equation (12) above the equation for the second derivative of the moment generating function with respect to the variable t is...

$$\frac{\delta^2}{\delta t^2}\,M_y(t)=\frac{\delta}{\delta\omega}\left(\frac{\delta}{\delta t}\,M_y(t)\right)\frac{\delta}{\delta t}\,\omega=\left(-\alpha-1\right)\alpha\,\theta\,\omega^{-\alpha-2}\times-\theta=\left(\alpha^2\theta^2+\alpha\,\theta^2\right)\omega^{-\alpha-2} \tag{13}$$

The Distribution Mean and Variance

Using Equation (12) above and setting $t=0$ the equation for the first moment of the distribution is...

$$\mathbb{E}\left[y\right]=\frac{\delta}{\delta t}\,M_y(t)=\alpha\,\theta\ \text{...when...}\ t=0\ \text{...such that...}\ \omega=1 \tag{14}$$

Using Equation (13) above and setting $t=0$ the equation for the second moment of the distribution is...

$$\mathbb{E}\left[y^2\right]=\frac{\delta^2}{\delta t^2}\,M_y(t)=\alpha^2\theta^2+\alpha\,\theta^2\ \text{...when...}\ t=0\ \text{...such that...}\ \omega=1 \tag{15}$$

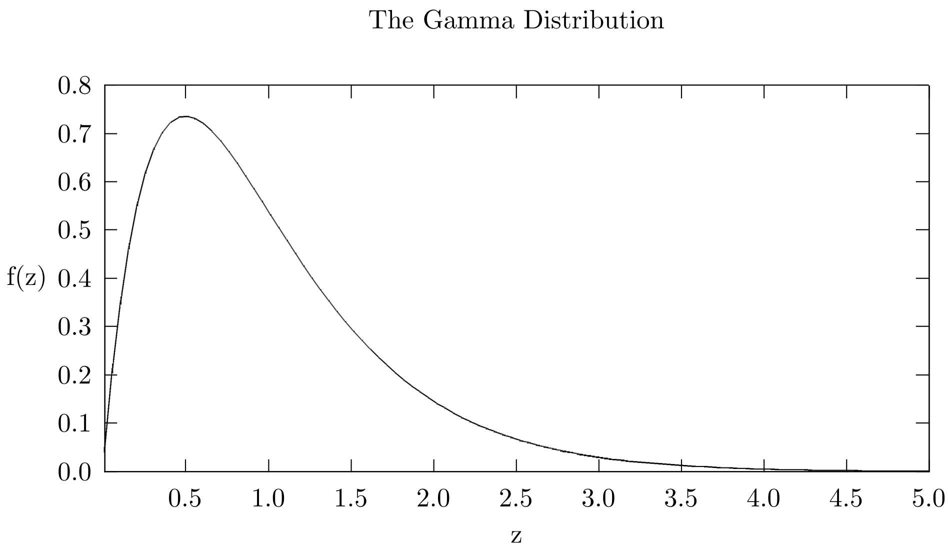
Using Equation (14) above the mean of the distribution is...

$$\text{mean}=\mathbb{E}\left[y\right]=\alpha\,\theta \tag{16}$$

Using Equations (14) and (15) above the mean of the distribution is...

$$\text{variance} = \mathbb{E}\Big[y^2\Big] - \Big[\mathbb{E}\Big[y\Big]\Big]^2 = \alpha\,\theta^2 \tag{17}$$

As an example the following graph charts the Gamma Distribution ($\alpha = 2$, $\theta = 0.50$) over the real number interval $[\alpha > 0, \alpha \leq 5]$...



Parameter Estimation

Assume that we are given the mean and variance of a sample of random variables assumed to be drawn from a Gamma Distribution. To work with the underlying distribution we need to match the first two moments of the sample data to the first two moments of the underlying Gamma Distribution. Given that we have the mean and variance of the sample data the first two moments of the sample data (S_1 and S_2, respectively) where the variable z_i is the i'th data point within that sample and n is the sample size are...

$$S_1 = \sum_{i=1}^{n} \frac{1}{n}\, z_i = \text{Sample mean} \;\; \text{...and...} \;\; S_2 = \sum_{i=1}^{n} \frac{1}{n}\, z_i^2 + \Bigg[\sum_{i=1}^{n} \frac{1}{n}\, z_i\Bigg]^2 = \text{Sample variance} + \text{mean squared} \tag{18}$$

We want to match the first two sample moments to the first two moments of the Gamma Distribution. Using Equations (14), (15) and (18) above the two simultaneous equations that must be solved are...

$$S_1 = \alpha\,\theta \;\; \text{...and...} \;\; S_2 = \alpha^2\theta^2 + \alpha\,\theta^2 \tag{19}$$

Using Appendix Equations (27) and (28) below the parameter estimates for the Gamma Distribution are...

$$\alpha = \frac{S_1^2}{S_2 - S_1^2} \;\; \text{...and...} \;\; \theta = \frac{S_2 - S_1^2}{S_1} \tag{20}$$

The Answer To Our Hypothetical Problem

Using the default data in Table 1 above the first two moments of the default rate distribution (S_1 and S_2, respectively) are...

$$S_1 = 0.01070 \;\; \text{...and...} \;\; S_2 = 0.01612^2 + 0.01070^2 = 0.00037 \tag{21}$$

The next step is to convert the default rate moments to number of defaults moments (M_1 and M_2, respectively) given a 200 bond portfolio. Using Equation (21) above the first two moments of the distribution of the number of bond defaults are...

$$M_1 = 200 \times S_1 = 2.14 \;\; \text{...and...} \;\; M_2 = 200^2 \times S_2 = 14.97 \tag{22}$$

Using Equations (20) and (22) above the parameters to the Gamma Distribution are...

$$\alpha=\frac{M_1^2}{M_2-M_1^2}=\frac{2.14^2}{14.97-2.14^2}=0.4406 \text{ ...and... } \theta=\frac{M_2-M_1^2}{M_1}=\frac{14.97-2.14^2}{2.14}=4.8571 \tag{23}$$

Using Equation (23) above and the Excel function GAMMA.DIST the answer to our hypothetical problem is...

$$\begin{aligned} P\Bigg[\text{Number Defaults} \geq 10\Bigg] &= \int\limits_{x=10}^{x=\infty} \frac{x^{\alpha-1}\,\text{Exp}\left\{-x\,\theta^{-1}\right\}}{\theta^{\alpha}\Gamma(\alpha)}\,\delta x \\ &= 1-\text{GAMMA.DIST}(10,\ 0.4406,\ 4.8571,\ \text{TRUE}) \\ &= 0.0352 \end{aligned} \tag{24}$$

The probability that there will be 10 or more bond defaults in any given year assuming no defaults at the beginning of that year is 3.52%.

Appendix

A. Using Equation (6) above the equation for the moment generating function of the Gamma Distribution is...

$$\begin{aligned} M_x(t) &= \int\limits_{x=0}^{x=\infty} f(\alpha,\theta)\,\text{Exp}\Bigg\{x\,t\Bigg\}\,\delta x \\ &= \frac{1}{\Gamma(\alpha)}\int\limits_{0}^{\infty}\theta^{-\alpha}\,x^{\alpha-1}\,\text{Exp}\Bigg\{-x\,\theta^{-1}\Bigg\}\,\text{Exp}\Bigg\{x\,t\Bigg\}\,\delta x \\ &= \frac{1}{\theta^{\alpha}\Gamma(\alpha)}\int\limits_{x=0}^{x=\infty} x^{\alpha-1}\,\text{Exp}\Bigg\{-x\Bigg(\theta^{-1}-t\Bigg)\Bigg\}\,\delta x \end{aligned} \tag{25}$$

B. Using Equations (7), (8) and (9) above the equation for the moment generating function of the Gamma Distribution post change of variables is...

$$\begin{aligned} M_y(t) &= \frac{1}{\theta^{\alpha}\Gamma(\alpha)}\int\limits_{y=0}^{y=\infty}\left(\frac{\theta}{1-\theta\,t}\right)^{\alpha-1} y^{\alpha-1}\,\text{Exp}\Bigg\{-y\Bigg\}\,\frac{\theta}{1-\theta\,t}\,\delta y \\ &= \frac{1}{\theta^{\alpha}\Gamma(\alpha)}\left(\frac{\theta}{1-\theta\,t}\right)^{\alpha}\int\limits_{y=0}^{y=\infty} y^{\alpha-1}\,\text{Exp}\Bigg\{-y\Bigg\}\,\delta y \\ &= \frac{1}{\Gamma(\alpha)}\left(\frac{1}{1-\theta\,t}\right)^{\alpha}\Gamma(\alpha) \\ &= \Bigg(1-\theta\,t\Bigg)^{-\alpha} \end{aligned} \tag{26}$$

C. Using Equation (19) above and solving for θ...

$$\text{if... } S_2=\alpha^2\theta^2+\alpha\,\theta^2 \text{ ...then... } S_2=S_1^2+\theta\,S_1 \text{ ...then... } \theta=\frac{S_2-S_1^2}{S_1} \tag{27}$$

Using Equations (19) and (27) above and solving for α...

$$\text{if... } S_1=\alpha\,\theta \text{ ...then... } \alpha=\frac{S_1}{\theta} \text{ ...then... } \alpha=\frac{S_1^2}{S_2-S_1^2} \tag{28}$$